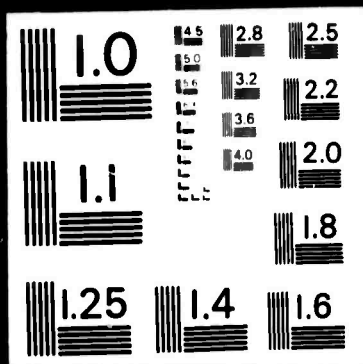MEMORANDUM
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ACOUSTIC-GRAVITY WAVES
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PREFACE

This report is part of RAND's continuing interest in the geo-
physical effects produced by nuclear explosions. Previous work
(RM-4225, RM-4388, RM-4494, RM-4858, RM-4946, RM-5616)[*] has been con-
cerned with the geomagnetic effects produced by such explosions. The
present study is the start of an investigation of some aspects of the
acoustic effects. This work was sponsored by the United States Air
Force and the Advanced Research Projects Agency.

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[*]Greifinger, C., P. Greifinger, Low-Frequency Hydromagnetic Waves in
the Ionosphere, The RAND Corporation, RM-4225, October 1964.

Greifinger, C., P. Greifinger, Transmission of Micropulsations
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pulsations through the Ionosphere and Lower Exosphere, The RAND
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Greifinger, C., P. Greifinger, Theory of Hydromagnetic Propagation
in the Ionospheric Waveguide, The RAND Corporation, RM-5616,
June 1968.

SUMMARY

A complete asymptotic analysis is carried out for the flow field produced by the instantaneous release of energy, at a point on the ground, in an isothermal atmosphere. A double integral representation of the flow is constructed from Laplace-Hankel transforms of the linearized equations. An asymptotic approximation to the integral is obtained by two successive applications of the method of stationary phase. It is found that there are three principal groups of dispersive waves behind the spherical acoustic front. One of these groups is contained in a high frequency band, and the other two in a low frequency band. The spatial domain of the low frequency waves is cut off by a front (caustic), the location of which is calculated. Some characteristics of the flow are discussed.

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I. INTRODUCTION

The problem of waves in a stratified atmosphere has received much attention and many different cases have been considered (cf Bibliography). The case which is analyzed here is one for which a complete asymptotic analysis can be carried out and an overall view of the resulting wave motion obtained. The principal simplification is that the waves are small disturbances to an isothermal atmosphere. The problem worked out in detail is the flow field produced by the instantaneous release of energy at a point in the ground plane (the analysis is easily extended to cases of energy release at a point above the ground). This case does not seem to have been discussed previously, although similar problems for monochromatic waves have been discussed.[1]

The basic method used for obtaining the asymptotic approximation is the repeated application of the method of stationary phase. The kinematic aspects of the groups of waves produced could have been based on the dispersion relation and the considerations of Whitham[2] and Lighthill.[3] But, it turned out to be more convenient for this paper to base all the calculations on the Fourier integrals derived by Laplace-Hankel transforms of the linearized equations. In any case there are three principal groups of waves, one in a high frequency band and two in a low frequency band. The spatial domain of the low frequency waves is cut off by a front (caustic). This phenomena was first noted by Mowbray and Rarity[4] in their study of dispersive waves in an incompressible stratified fluid. A similar phenomenon appears in the pattern of waves around a moving ship.

In Section II the basic equations are given and the linearization is carried out in suitable dimensionless units. The basic small parameter is shown to be the ratio of the energy released to the internal energy in a volume of scale height dimensions. In addition, due to the exponential decay of density, there is an exponential growth of disturbances unfavorable for the validity of the theory. However, there should still be a considerable range in which the approximation is quantitatively correct as well as qualitatively instructive.

In Section III the application of Laplace transform in time and Hankel transform with respect to cylindrical radius enables an integral representation of the exact solution to be obtained. In Section IV it is shown how the acoustic spherical wave front is contained in the representation. Non-linear effects are probably most important in describing the amplification of the front into a shock wave and its subsequent propagation. However, the main residual disturbance is a system of dispersive waves behind the front. By studying the double integral representation, it is seen that a cylindrical ground wave exists in addition to propagating waves in low and high frequency bands. Section V studies these propagating dispersive groups and presents the final description of the flow as well as some numerical results.

II. BASIC EQUATIONS

We consider motion in an isothermal atmosphere above a ground plane produced by instantaneous energy release at a point on the ground, the origin in Fig. 2.1. The equilibrium atmosphere is thus characterized by the usual exponential distributions with scale height h.

$$\frac{P_o(z)}{P^*} = e^{-\frac{z}{h}} = \frac{\rho_o(z)}{\rho^*} \tag{2.1}$$

P^*, ρ^*, T^* = sea level ($z = 0$) pressure, density, temperature, h = scale height = $\frac{RT^*}{g}$. Note that $c^* = \sqrt{\gamma}\, c_g$ where c^* = isentropic sound speed = $\sqrt{\gamma RT^*}$, c_g = gravity wave speed = $\sqrt{gh}$. The dimensionless equations of motion, continuity, momentum, entropy for a perfect gas are

$$\frac{\partial \rho}{\partial t} + \text{div } \rho \vec{q} = 0 \tag{2.2}$$

$$\frac{\partial \vec{q}}{\partial t} + \vec{q} \cdot \vec{\nabla} \vec{q} = -\frac{1}{\gamma \rho} \nabla P - \frac{1}{\gamma} \vec{k} \tag{2.3}$$

$$(\frac{\partial}{\partial t} + \vec{q} \cdot \nabla) \frac{P}{\rho^\gamma} = \frac{1}{\rho^\gamma} \epsilon\ \delta(x)\delta(y)\delta(z)\delta(t) \ . \tag{2.4}$$

Here lengths have been referred to the scale height h, velocities to c^*, times to h/c^*, pressure to P^*, and density to ρ^*. The basic parameter of the problem is

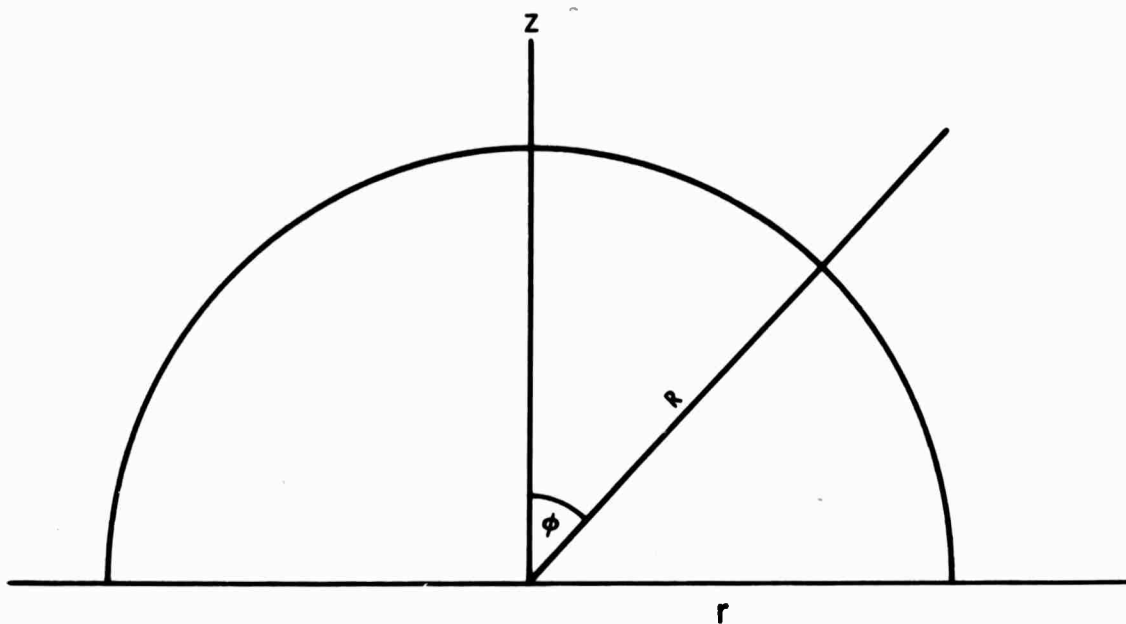$$\epsilon = \frac{(\gamma-1)Q_o}{h^3 P^*} \tag{2.5}$$

Fig. 2.1 - Co-ordinate system, with origin at the point
of energy release.

where Q_o = energy released at $t = 0$; ϵ roughly measures the energy release compared to the internal energy stored in a scale height volume. The initial conditions at $t = 0$ are

$$P = \rho = e^{-z}, \quad \vec{q} = 0, \tag{2.6a}$$

the boundary conditions at $z = 0$ is

$$q_z = 0. \tag{2.6b}$$

ACOUSTIC EXPANSION

The acoustic expansion is based on $\epsilon \ll 1$ and represents the flow as small changes superimposed on the ambient state:

$$\vec{q} = \epsilon \vec{u} + \ldots, \qquad \vec{u} = (u,w) \tag{2.7}$$
$$\text{where } u = \text{radial component}$$
$$w = \text{vertical component}$$

$$P = e^{-z} \{1 + \epsilon p + \ldots\} \tag{2.8}$$

$$\rho = e^{-z} \{1 + \epsilon\sigma + \ldots\} . \tag{2.9}$$

The equations of order ϵ that result from (2.2, 2.3, 2.4) are thus, for the case of axial symmetry,

$$\sigma_t + u_r + \frac{u}{r} + w_z - w = 0 \qquad r = \sqrt{x^2 + y^2} \tag{2.10}$$

$$u_t \qquad\qquad\qquad = -\frac{1}{\gamma} P_r \tag{2.11}$$

$$w_t \qquad\qquad\qquad = -\frac{1}{\gamma} P_z + \frac{p-\sigma}{\gamma} \tag{2.12}$$

$$P_t - \gamma \sigma_t + (\gamma-1)w = \frac{1}{2\pi} \frac{\delta(r)}{r} \delta(z)\delta(t) . \tag{2.13}$$

The initial and boundary conditions for the perturbations are

$$p = \sigma = \vec{u} = 0 \quad \text{at} \quad t = 0- \tag{2.14}$$

$$w = 0 \text{ at } z = 0, \ t > 0 \tag{2.15}$$

and the main wave motion is assumed outgoing. The characteristic surfaces of the system (2.10, -13) are given by

$$\phi_t^2 = 0, \ \phi_t^2 - \phi_z^2 - \phi_r^2 = 0 \tag{2.16}$$

where $\phi(r,z,t)$ = const. on a characteristic surface in (r,z,t) space. The first factor in (2-16) represents possible discontinuity surfaces along fixed cylinders $r(z)$ in space while the second shows the possibility of a spherical acoustic front (the sound speed is constant everywhere in this model).

III. FORMAL SOLUTIONS

A formal integral representation of the solution is constructed by a combination of Laplace and Hankel transformation. Let

$$\bar{g}(s) = \int_0^\infty e^{-st} g(t)\, dt \qquad \text{(Laplace transform)}^* \qquad (3.1)$$

so that the system (2.10,13) has $\frac{\partial}{\partial t} \to s$, with the zero initial conditions.

$$s\bar{\sigma} + \bar{u}_r + \frac{\bar{u}}{r} + \bar{w}_z - \bar{w} = 0 \qquad (3.2)$$

$$s\bar{u} + \frac{1}{\gamma} \bar{p}_r = 0 \qquad (3.3)$$

$$\frac{1}{\gamma} \bar{\sigma} + s\bar{w} + \frac{1}{\gamma} (\bar{p}_z - \bar{p}) = 0 \qquad (3.4)$$

$$- \gamma s\bar{\sigma} + (\gamma-1)\bar{w} + s\bar{p} = \frac{1}{2\pi} \frac{\delta(r)}{r} \delta(z). \qquad (3.5)$$

Using the Hankel transforms

$$f(r) = \int_0^\infty F(\omega)\, J_\nu(\omega r)\, \omega d\omega, \quad F(\omega) = \int_0^\infty f(r)\, J_\nu(\omega r)\, r dr \quad \nu > (-\tfrac{1}{2})$$

$$(3.6)$$

note that

$$\frac{\delta(r)}{2\pi r} = \int_0^\infty \widetilde{\delta}(\omega)\, J_0(\omega r)\, \omega d\omega, \qquad \widetilde{\delta}(\omega) = \int_0^\infty \frac{\delta(r)}{2\pi r}\, J_0(\omega r)\, r dr = \frac{1}{2\pi}.$$

$$(3.7)$$

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*We intend to use the usual inversion formula

$$g(t) = 1/2\pi i \int_{-i\infty}^{i\infty} e^{st}\, \bar{g}(s)\, ds \ .$$

Thus, let

$$(\bar{\sigma}, \bar{w}, \bar{p}) = \int_0^\infty (R, W, P) \, J_0(kr) \, kdk \tag{3.8}$$

$$\bar{u} = \int_0^\infty U(k) \, J_1(kr) \, kdk \ . \tag{3.9}$$

The choices in (3.8,9) are made consistent with the overall symmetry of the equations and insure that $\lim_{r \to 0} ru_r \to 0$, so that the behavior at the axis of symmetry is regular. Using the properties of J_0, J_1, etc. the basic system, (3.2,5) becomes

$$sR + kU + \frac{dW}{dz} - W = 0 \tag{3.10}$$

$$sU - \frac{k}{\gamma} P = 0 \tag{3.11}$$

$$\frac{1}{\gamma} R + sW + \frac{1}{\gamma} (\frac{dP}{dz} - P) = 0 \tag{3.12}$$

$$- \gamma sR + (\gamma-1)W + sP = \frac{1}{2\pi} \delta(z) \ . \tag{3.13}$$

This system is linear and second order with respect to z, so that there are two basic roots λ_1, λ_2, for solutions of the form $(R,U,W,P) \sim e^{\lambda z}$. The characteristic determinant from (3.10,-13) is

$$\begin{vmatrix} s & k & \lambda-1 & 0 \\ 0 & s & 0 & -k/\gamma \\ \frac{1}{\gamma} & 0 & s & \frac{1}{\gamma}(\lambda-1) \\ -\gamma s & 0 & \gamma-1 & s \end{vmatrix} = 0 \tag{3.14}$$

and the characteristic equation is

$$(\lambda-1)^2 + (\lambda-1) - \left\{s^2 + k^2 (1 + \frac{\gamma-1}{\gamma^2+s^2})\right\} = 0 \quad . \tag{3.15}$$

The basic roots are

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + s^2 + k^2(1 + \frac{\gamma-1}{\gamma^2 s^2})} \quad . \tag{3.16}$$

For definiteness the contours in the ⑤ plane is at first chosen so that Re $\sqrt{\ } \geq 0$ in (3.16).

In order to formulate "jump" conditions it is most convenient to eliminate (R,U) from the system (3.10-13) and obtain

$$\gamma \frac{dW}{dz} - W + (\frac{k^2}{s} + s)P = \frac{1}{2\pi} \delta(z) \tag{3.17}$$

$$\left\{\gamma^2 s^2 + (\gamma-1)\right\} W + \gamma s (\frac{dP}{dz} - P) + sP = \frac{1}{2\pi} \delta(z) \quad . \tag{3.18}$$

Integration of (3.17) from (0- to 0+) yields the jump conditions

$$[W]_{z=0} = \frac{1}{2\pi\gamma} \ , \quad [P]_{z=0} = \frac{1}{2\pi\gamma} \frac{1}{s} \quad . \tag{3.19}$$

Since the solution should vanish ahead of the acoustic wave front ($\sqrt{r^2+z^2} > t$) only the root λ_2 of (3.16) need be considered and we write

$$\lambda = \frac{1}{2} - \mu(s,k;\gamma) \text{ where } \mu = \sqrt{\frac{1}{4} + s^2 + k^2(1 + \frac{\beta^2}{s^2})} \ ; \tag{3.20}$$

and

$$\beta^2 = \frac{\gamma-1}{\gamma^2} \quad .$$

Note that $0 < \beta^2 < \frac{1}{2}$; ($\gamma = 1.4$, $\beta^2 = .22$) for realistic values of γ. Since $W = 0$, for $z = 0-$, the jump condition (3.19) provides the boundary conditions for W as $z \to 0+$ and the solution for W is

$$W(z;s,k) = \frac{1}{2\pi\gamma} e^{\frac{1}{2}z} e^{-\mu z} . \qquad (3.21)$$

It follows from (3.17), for $z > 0$, that

$$P(z;s;k) = \frac{1}{2\pi} \frac{\frac{2-\gamma}{2\gamma} + \mu}{s^2 + k^2} s . \qquad (3.22)$$

Applying the double inversion of Laplace and Hankel transforms, we obtain a formal representation of the perturbation pressure field

$$e^{-\frac{1}{2}z} p(r,z,t) = \frac{1}{2\pi} \int_0^\infty J_0(kr) F(k;z,t) \, k \, dk \qquad (3.23)$$

where

$$F(k;z,t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st - \mu(s,k)z} \frac{\frac{2-\gamma}{2\gamma} + \mu}{s^2 + k^2} s \, ds . \qquad (3.24)$$

The integration of (3.24) is regarded as carried out first, for a fixed real k. This integration over s is carried out, at first, along a path parallel to the imaginary axis to the right of all singularities in the $\circledS$ plane, so that the initial conditions are satisfied. The $\circledS$ plane is cut suitably so that $\text{Re}(\mu) \geq 0$ on the contour.

The pressure field is the main quantity of interest but all the other quantities easily follow from the system (3.10-13).

IV. APPROXIMATE AND ASYMPTOTIC EVALUATIONS

In this note the main interest is centered on the dispersive waves behind the acoustic front. This is the main disturbance which contains eventually most of the energy put in at $t = 0$.

However, first, a crude approximation valid near the acoustic wave front comes from considering large (k,s) in (3.23), to obtain

$$p(r,z,t) \doteq e^{\frac{1}{2}z} \frac{1}{4\pi^2 i} \int_{-i\infty}^{i\infty} s \, e^{st} ds \int_{o}^{\infty} \frac{e^{-z\sqrt{s^2+k^2}}}{\sqrt{s^2+k^2}} J_0(kr) k dk \; . \quad (4.1)$$

The second integral is a special case of an integral evaluated in Watson's Bessel functions.[5]

$$\int_{o}^{\infty} \frac{e^{-z\sqrt{s^2+k^2}}}{\sqrt{s^2+k^2}} J_0(kr) k dk = \frac{e^{-s\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}} \quad (4.2)$$

so that (4.1) is

$$p(r,z,t) \doteq \frac{e^{\frac{1}{2}z}}{2\pi R} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} s \, e^{s(t-R)} ds \quad (4.3)$$

where $R = \sqrt{r^2+z^2}$ = spherical radius. According to the usual interpretation the integral in (4.3) is the derivative of a delta function, or

$$p(r,z,t) \doteq e^{\frac{1}{2}z} \frac{1}{2\pi R} \delta'(t-R) \quad (4.4)$$

$\delta'(t-R)$ is the acoustic version of the compression followed by an expansion that is felt near the wave front. The intensity is

doubled, due to the ground plane, over that of a point source in free space. Further, the factor $e^{\frac{1}{2}z}$ represents the amplification of the waves proportional to $\rho_o(z)^{-\frac{1}{2}}$, according to the usual local energy considerations.

Asymptotic integration of (3.23,24) which is valid in the main wave zone for large (r,t) can be carried out by repeated applications of the method of stationary phase coupled with the use of an asymptotic representation of J_o.

In order to give definite meaning to the integral the following plan is adopted. The ⑤ plane is suitably cut and the integral of (3.24) for $F(k;z,t)$ is expressed as an integral along segments of the imaginary axis $(s=i\omega)$. That is, it is expressed as a Fourier integral over the frequencies ω which, for a given k, can propagate. In doing this, the residues at certain poles (due to presence of the ground), have to be evaluated. Next, in the real double integrals, (ω,k), the order of integration is changed, since the stationary phase approximation can more easily be carried out in the k-integral. However, this can only be done if the asymptotic approximation for J_o is used. After this is completed, the answer is expressed as a Fourier integral. This integral is again of such a form that the method of stationary phase can be used and its qualitative features are easily discussed. (See 5.) In addition some quantitative results are also presented. Some details of the procedure just outlined are now given.

First note that the exponent in (3.24)

$$\mu(s,k) = \frac{1}{s}\left\{ s^4 + (\frac{1}{4} + k^2)s^2 + \beta^2 k^2 \right\}^{\frac{1}{2}} \tag{4.5}$$

can be written as

$$\mu = \frac{1}{s}\left\{ (s^2 + \omega_1^2(k))(s^2 + \omega_2^2(k)) \right\}^{\frac{1}{2}} \tag{4.6}$$

so that the integrand has branch points at $s = \pm i\omega_1, \pm i\omega_2$ where

$$\omega_2(k) = \frac{1}{2}\left[(\frac{1}{4} + k^2 + 2\beta k)^{\frac{1}{2}} + (\frac{1}{4} + k^2 - 2\beta k)^{\frac{1}{2}}\right] \qquad (4.7)$$

$$\omega_1(k) = \frac{1}{2}\left[(\frac{1}{4} + k^2 + 2\beta k)^{\frac{1}{2}} - (\frac{1}{4} + k^2 - 2\beta k)^{\frac{1}{2}}\right] \qquad (4.8)$$

ω_2 is a lower cut-off frequency, ω_1 an upper cut-off frequency. (See Fig. 4.1) In addition, the integrand of (3.24) has poles at $s = \pm ik$, and an essential singularity (due to μ in the exponent) at the origin. It can be verified that the integrand is one-valued in the entire plane with barriers as shown, if the principal branches of the angles associated with each branch point are chosen as shown in Fig. 4.2. The picture is drawn for a given k and is qualitatively the same for all k. Note the arguments of μ at the various points on the imaginary axis.

$$\arg \mu_A = \arg \mu_D = \arg \mu_E = \arg \mu_H = \pi/2 \qquad (4.9)$$

$$\arg \mu_B = \arg \mu_C = \arg \mu_F = \arg \mu_G = -\pi/2 . \qquad (4.10)$$

The application of Cauchy's theorem enables the integral of F on the original path to be expressed as integrals on segments of the imaginary axis as shown, plus $(2\pi i)$ x (sum of the residues) at the poles $(s = \pm ik)$. This is true since the integral on a large circle in the left half plane vanishes $(Re(s) < 0)$. Also, the essential singularity at the origin is outside the contour used for Cauchy's theorem, and thus does not contribute a residue. Thus

$$F(k;z,t) = (\sum_{s = \pm ik} Res) + F_1 + F_2 + F_3 + F_4 + F_5 + F_6$$
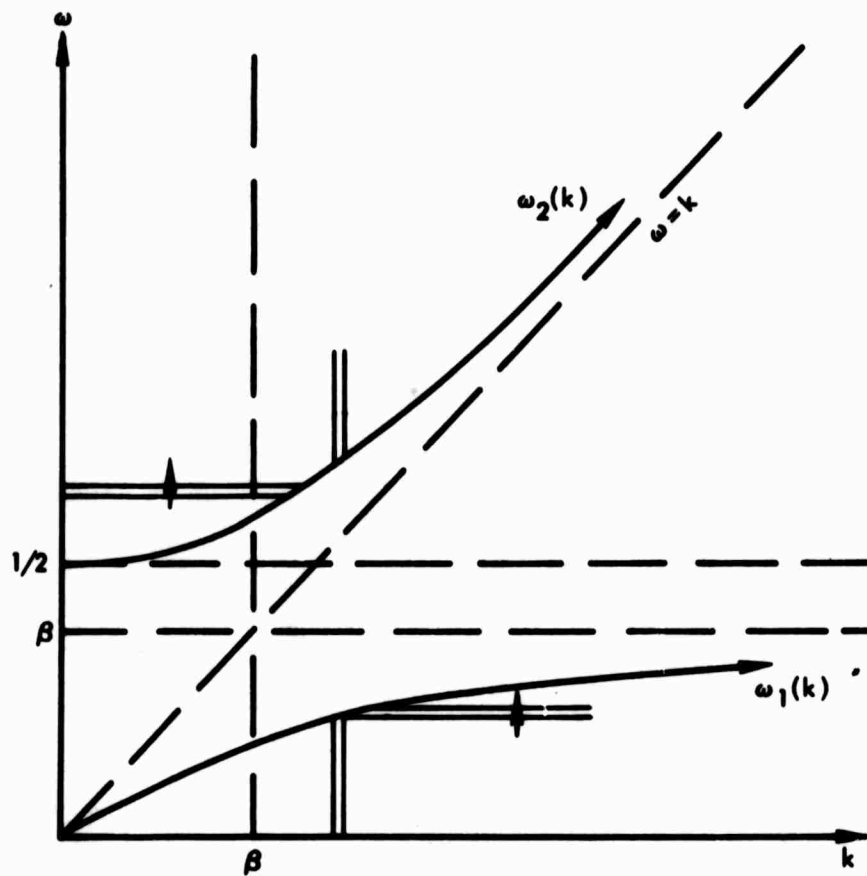
$$(4.11)$$

Fig. 4.1 - Location of the cutoff frequencies, given by
        Eqs. 4.7 and 4.8, which arise from the change
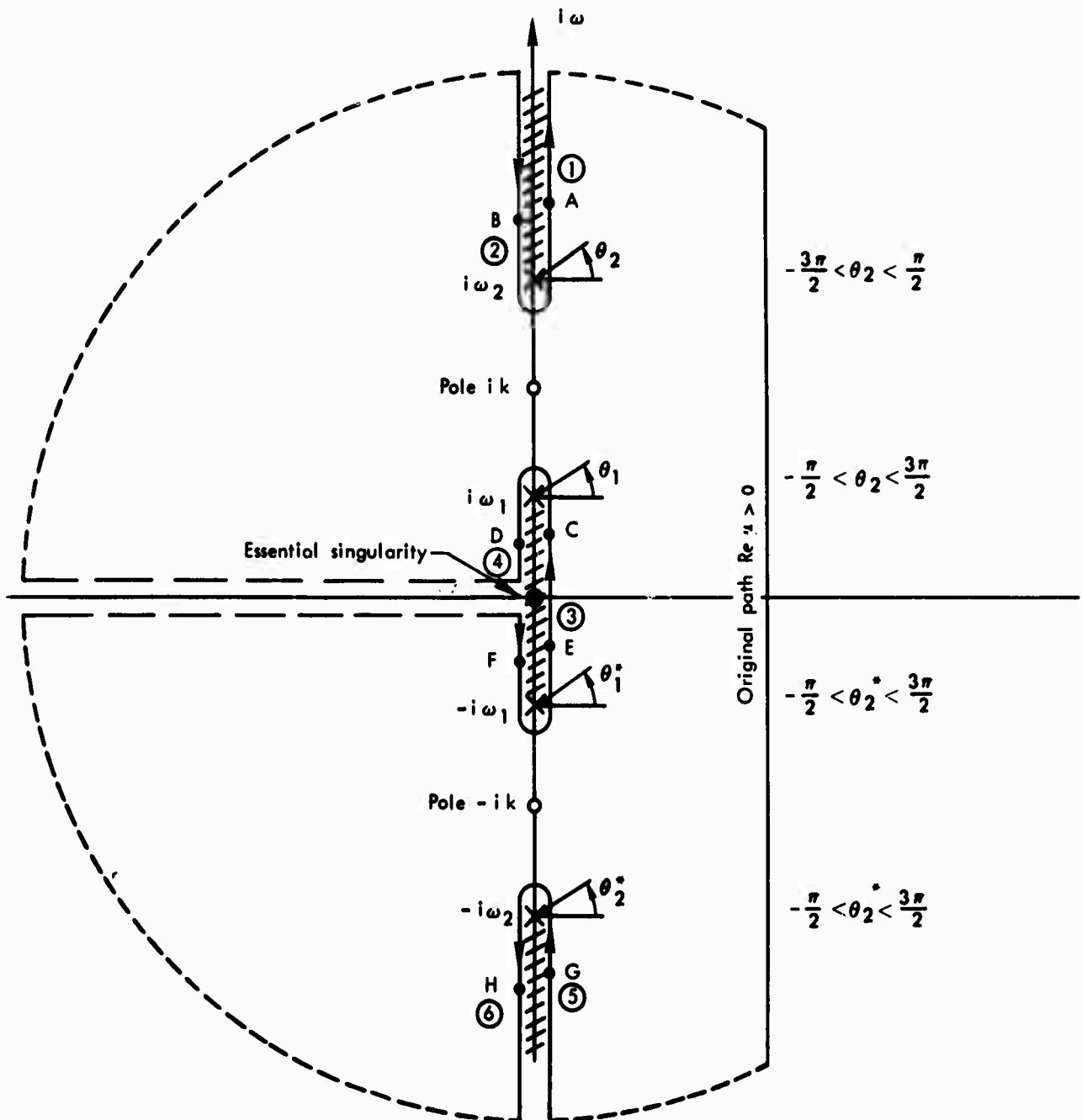   in the order of integration of Eq. 4.1.

Fig. 4.2 - Contour for carrying out the integration in the
s-plane, with the location of poles, branch
points, and essential singularity as indicated.

where

$$F_1(k) = \frac{1}{2\pi i} \int_{\omega_2(k)}^{\infty} e^{i(\omega t - \mu_1 z)} \frac{(\frac{2-\gamma}{2\gamma} + i\mu_1)}{\omega^2 - k^2} \omega \, d\omega \qquad (4.12)$$

$$F_2(k) = \frac{1}{2\pi i} \int_{\omega_0}^{\omega_2} e^{i(\omega t + \mu_1 z)} \frac{(\frac{2-\gamma}{2\gamma} - i\mu_1)}{\omega^2 - k^2} \omega \, d\omega \qquad (4.13)$$

and $\mu = i\mu_1$ on 1 , etc.

$$\mu_1 = \frac{1}{\omega} \sqrt{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2)} = \frac{1}{\omega} \sqrt{\omega^4 - (\frac{1}{2} + k^2)\omega^2 + \beta^2 k^2} \qquad (4.14)$$

$$F_5(k) = \frac{1}{2\pi i} \int_{-\infty}^{-\omega_2} e^{i(\omega t + \mu_5 z)} \frac{\frac{2-\gamma}{2\gamma} - i\mu_5}{\omega^2 - k^2} \omega \, d\omega \qquad (4.15)$$

$$F_6(k) = \frac{1}{2\pi i} \int_{-\omega_z}^{-\infty} e^{i(\omega t - \mu_5 z)} \frac{\frac{2-\gamma}{2\gamma} + i\mu_5}{\omega^2 - k^2} \omega \, d\omega \qquad (4.16)$$

Noting that $\mu_5 = -\mu_1$ these integrals can be combined to

$$F_1 + F_2 + F_5 + F_6 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\omega \, d\omega}{\omega^2 - k^2} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_1) e^{i(\omega t - \mu_1 z)} - (\frac{2-\gamma}{2\gamma} - i\mu_1) e^{i(\omega t + \mu_1 z)} \right\}$$

$$|\omega| > \omega_2(k) . \qquad (4.17)$$

The integration is carried out in (4.17) only over those frequencies $|\omega| > \omega_2(k)$ which can propagate as waves. In a similar way,

$$F_3 + F_4 = \frac{1}{2\pi i} \int_{-\omega_1}^{\omega_1} \frac{\omega \, d\omega}{k^2 - \omega^2} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_1) e^{i(\omega t - \mu_1 z)} - (\frac{2-\gamma}{2\gamma} - i\mu_1) e^{i(\omega t + \mu_1 z)} \right\} .$$

$$(4.18)$$

Thus, all these integrals can be combined to a Fourier integral on only those frequencies which can propagate.

$$F_1 + F_2 + F_3 + F_4 + F_5 + F_6 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\omega \, d\omega}{|\omega^2 - k^2|} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_1) e^{i(\omega t - \mu_1 z)} - (\frac{2-\gamma}{2\gamma} - i\mu_1) e^{i(\omega t + \mu_1 z)} \right\}$$

$$|\omega| < \omega_1, \ |\omega| > \omega_2 . \qquad (4.19)$$

Further, the residue at $s = \pm ik$ is easily calculated:

$$(\mu^2(ik) = \zeta - \beta^2 = (\frac{2-\gamma}{2\gamma})^2)$$

$$\text{Res } (s = \pm ik) = \frac{2-\gamma}{2\gamma} e^{-\frac{2-\gamma}{2\gamma} z} e^{\pm ikt} . \qquad (4.20)$$

Thus

$$\sum_{s = \pm ik} \text{Res} = \frac{2-\gamma}{\gamma} e^{-\frac{2-\gamma}{2\gamma} z} \cos kt . \qquad (4.21)$$

The pressure field is calculated by applying the Hankel transform to (4.19) and (4.21) as in (3.23). The part produced by the residue at the poles (4.21) is called the ground wave P_G and thus has the representation

$$e^{-\frac{1}{2}z} \ p_G = \frac{1}{2\pi} \ \frac{2-\gamma}{\gamma} \ e^{-(\frac{2-\gamma}{2\gamma})z} \ \int_0^\infty J_0(kr) \cos kt \ k \ dk \ . \qquad (4.22)$$

The integral in (4.22) is divergent, but can be interpreted as

$$\frac{\partial}{\partial t} \int_0^\infty \sin kt \ J_0(kr) \ dk = -\frac{t}{(t^2-r^2)^{3/2}} \ . \qquad (4.23)$$

Then,

$$e^{-\frac{1}{2}z} \ p_G = \frac{1}{2\pi} \ (\frac{2-\gamma}{\gamma}) \ e^{-\frac{2-\gamma}{2\gamma}z} \ \left(\frac{-t}{(t^2-r^2)^{3/2}}\right) \qquad (4.24)$$

(4.24) represents a cylindrical wave, which exists only behind the spherical front. It is an exact solution to the original equations as given in Lamb[6] (p. 548). The wave is excited from the axis (r=0) by the vertical passage of the spherical acoustic wave. The remainder of the pressure field comes from the Hankel transform of (4.19). The order of integration is changed and the integral is broken up into low and high frequency bands (cf. Fig. 3.1)

$$p = p_G + p_I + p_{II} \qquad (4.25)$$

$$e^{-\frac{1}{2}z} p_I = \frac{1}{2\pi i} \int_{-\beta}^{\beta} \omega \ d\omega \ \frac{1}{2\pi} \int_{k_1(\omega)}^{\infty} \frac{J_0(kr)k}{k^2-\omega^2} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_1)e^{i(\omega t-\mu_1 z)} \right.$$

$$\left. - (\frac{2-\gamma}{2\gamma} - i\mu_1) \ e^{i(\omega t+\mu_1 z)} \right\} dk \qquad (4.26)$$

$$e^{-\frac{1}{2}z}P_{II} = \frac{1}{2\pi i} \int_{-\infty \atop |\omega|>\frac{1}{2}}^{\infty} \omega \, d\omega \, \frac{1}{2\pi} \int_{k_1(\omega)}^{\infty} \frac{J_0(kr)k}{\omega^2-k^2} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_1)e^{i(\omega t-\mu_1 z)} \right.$$

$$\left. - (\frac{2-\gamma}{2\gamma} - i\mu_1)e^{i(\omega t+\mu_1 z)} \right\} dk \, . \tag{4.27}$$

Here

$$k_1(\omega) = |\omega| \sqrt{\frac{\omega^2-\frac{1}{4}}{\omega^2-\beta^2}}$$

from (4.14). These integrals can be expressed as integrals over positive frequencies in low and high frequency bands.

$$e^{-\frac{1}{2}z}P_I = \frac{1}{\pi} \text{Im} \int_0^\beta \omega e^{i\omega t} \, G_I(\omega) d\omega \tag{4.28}$$

$$e^{-\frac{1}{2}z}P_{II} = \frac{1}{\pi} \text{Im} \int_{\frac{1}{2}}^\infty \omega e^{i\omega t} \, G_{II}(\omega) d\omega \tag{4.29}$$

where

$$G_I(\omega) = \frac{1}{2\pi} \int_{k_1}^\infty \frac{J_0(kr)k}{k^2-\omega^2} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_1)e^{-i\mu_1 z} - (\frac{2-\gamma}{2\gamma} - i\mu_1)e^{i\mu_1 z} \right\} dk$$

$$\tag{4.30}$$

$$G_{II}(\omega) = \frac{1}{2\pi} \int_0^{k_1} \frac{J_0(kr)k}{\omega^2-k^2} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_1)e^{-i\mu_1 z} - (\frac{2-\gamma}{2\gamma} - i\mu_1)e^{i\mu_1 z} \right\} dk \, .$$

$$\tag{4.31}$$

The asymptotic formula

$$J_0(kr) \cong \sqrt{\frac{2}{\pi kr}} \cos(kr - \frac{\pi}{4}) = \frac{1}{\sqrt{2\pi kr}} \left\{ e^{ikr - i\frac{\pi}{4}} + e^{-ikr + i\frac{\pi}{4}} \right\}$$

(4.32)

is now used in (4.30, 4.31). This approximation is very good for r
large except for those k near zero (in G_{II}). But for k close to
zero the integrand is small anyway so that the approximation of (4.32)
is probably excellent. The apparent singularity at (r=0) in (4.32)
in fact disappears from the answer later. Thus,

$$G_I \cong \frac{1}{(2\pi)^{3/2}\sqrt{r}} \int_{k_1(\omega)}^{\infty} \frac{\sqrt{k}\,dk}{k^2 - \omega^2} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_1)(e^{i\phi_1(k)} + e^{i\phi_1(-k)}) \right.$$

$$\left. - (\frac{2-\gamma}{2\gamma} - i\mu_1)(e^{i\phi_2(k)} + e^{i\phi_2(-k)}) \right\}$$

(4.33)

$$G_{II} \cong \frac{1}{(2\pi)^{3/2}\sqrt{r}} \int_{0}^{k_1(\omega)} \frac{\sqrt{k}\,dk}{\omega^2 - k^2} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_1)(e^{i\phi_1(k)} + e^{i\phi_1(-k)}) \right.$$

$$\left. - (\frac{2-\gamma}{2\gamma} - i\mu_1)(e^{i\phi_2(k)} + e^{i\phi_2(-k)}) \right\} .$$

(4.34)

Here

$$\phi_1(k) = kr - \mu_1(k)z - \frac{\pi}{4}\,\text{sign}\,k \qquad \phi_1(-k) = -kr - \mu_1 z + \frac{\pi}{4}$$

(4.35)

$$\phi_2(k) = kr + \mu_1 z - \frac{\pi}{4}\,\text{sign}\,k \qquad \phi_2(-k) = -kr + \mu_1 z + \frac{\pi}{4}$$

$$\mu_1 = \frac{1}{\omega} \sqrt{(\beta^2-\omega^2)k^2 - \omega^2(\tfrac{1}{4} - \omega^2)} = \frac{1}{\omega}\sqrt{\omega^2(\omega^2-\tfrac{1}{4}) - (\omega^2-\beta^2)k^2} \; .$$

(4.36)

Note that

$$\frac{d\mu_1}{dk} = \frac{\beta^2-\omega^2}{\omega} \; \frac{k}{[(\beta^2-\omega^2)k^2+\omega^2(\omega^2-\tfrac{1}{4})]^{\frac{1}{2}}} = \frac{\beta^2-\omega^2}{\omega^2} \; \frac{k}{\mu_1} \quad \begin{array}{l} > 0 \text{ in } G_I \\[4pt] < 0 \text{ in } G_{II} \end{array}$$

(4.37)

$$\frac{d^2\mu_1}{dk^2} = -\omega \; \frac{(\tfrac{1}{4}-\omega^2)(\beta^2-\omega^2)}{[(\beta^2-\omega^2)k^2 + \omega^2(\omega^2-\tfrac{1}{4})]^{3/2}} < 0, \text{ always.} \qquad (4.38)$$

Now, the method of stationary phase (Ref. 6, p. 395) is applied to the integrals in (4.33) and (4.34) with the rapidly oscillating exponents $\emptyset_{1,2}(\pm k)$. The main contribution comes from wave numbers where $\emptyset'(k_s) = 0$ (main group). Details of the calculation are shown only for G_I. The point of stationary phase in $\emptyset_1(k)$ is given by

$$\frac{d\mu_1}{dk} = \frac{r}{z} \; . \qquad (4.39)$$

Since $\dfrac{d\mu_1}{dk} > 0$ in G_I , only $\emptyset_1(k)$, $\emptyset_2(-k) = -\emptyset_1(k)$ have stationary points and contribute the dominant terms to the answer. Thus

$$(\frac{d\mu_1}{dk})^2 = \frac{r^2}{z^2} = \frac{(\beta^2-\omega^2)k_s^2}{\omega^2[(\beta^2-\omega^2)k_s^2+\omega^2(\omega^2-\tfrac{1}{4})]}$$

or

$$k_s = \omega^2 \sqrt{\frac{\omega^2-\tfrac{1}{4}}{\omega^2-\beta^2}} \; \frac{\sin\varphi}{\sqrt{\omega^2-\beta^2\cos^2\varphi}} \; . \qquad (4.40)$$

Here φ is the pole angle in physical space, $r = R\sin\varphi$, $z = R\cos\varphi$. According to (4.40), a stationary point occurs in G_I only for those frequencies $\omega > \beta\cos\varphi$. According to the method, the integrand is evaluated at the stationary point except for the exponent which is approximated as

$$\emptyset_1(k) = \emptyset_1(k_s) + \frac{(k-k_s)^2}{2}\ \emptyset_1''(k_s) \tag{4.41}$$

(subscript s denotes quantities evaluated at the stationary point). Due to the rapid oscillation away from $k-k_s$, the limits of integration can be extended to ∞ without introducing a significant error. Note the formulas

$$\omega^2 - k_s^2 = \frac{\omega^2\ D(\omega,\varphi)}{(\omega^2-\beta^2)(\omega^2-\beta^2\cos^2\varphi)} \tag{4.42}$$

where

$$D(\omega,\varphi) = (\omega^2-\beta^2)^2\cos^2\varphi + (\tfrac{1}{2}-\beta^2)\omega^2\ \sin^2\varphi > 0 \tag{4.43}$$

$$\mu_s = \cos\varphi\sqrt{\frac{(\tfrac{1}{2}-\omega^2)(\beta^2-\omega^2)}{\omega^2-\beta^2\cos^2\varphi}} \tag{4.44}$$

$$\emptyset_{1s} = R\sqrt{\frac{\tfrac{1}{2}-\omega^2}{\beta^2-\omega^2}}\ \sqrt{\omega^2-\beta^2\cos^2\varphi} - \frac{\pi}{4} = \emptyset_{2s}\ . \tag{4.45}$$

For G_{II}, the only stationary points occur in $\emptyset_2(k)$ and $\emptyset_1(-k)$. Thus, the stationary phase approximation yields,

$$G_I \cong \frac{1}{(2\pi)^{3/2}\sqrt{r}}\ \frac{\sqrt{k_s}}{k_s^2-\omega^2}\ 2i\ \mathrm{Im}\ \left\{(\frac{2-\gamma}{2\gamma}+i\mu_s)e^{i\emptyset_{1s}}\int_{-\infty}^{\infty}e^{\frac{i}{2}(k-k_s)^2\emptyset_{1s}''}\ dk\right\}$$

$$\tag{4.46}$$

$$G_{II} \cong \frac{1}{(2\pi)^{3/2}\sqrt{r}} \quad \frac{\sqrt{k_s}}{\omega^2-k_s^2} \; 2i \; \text{Im} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_s)e^{-i\emptyset_{2s}} \int_{-\infty}^{\infty} e^{-\frac{i}{2}(k-k_s)^2\emptyset''_{1s}} \; dk \right\} .$$

(4.47)

The integrals in (4.46), (4.47) are

$$\int_{-\infty}^{\infty} e^{\frac{i}{2}(k-k_s)^2\emptyset''_{1s}} \; dk = \frac{\sqrt{2\pi}}{\sqrt{\emptyset''_{1s}}} \; e^{i\frac{\pi}{4}} \int (\text{since } \emptyset''_{1s} = -z\mu''_{1s} > 0)$$

$$\int_{-\infty}^{\infty} e^{-\frac{i}{2}(k-k_s)^2\emptyset''_{2s}} \; dk = \frac{\sqrt{2\pi}}{\sqrt{|\emptyset''_{2s}|}} \; e^{i\frac{\pi}{4}} \int (\text{since } \emptyset''_{2s} = z\mu''_{1s} < 0) .$$

Using the expressions at the stationary point tabulated above, we obtain for (4.46, 4.47)

$$G_I \cong \frac{\cos\varphi}{2\pi R} \; \frac{(\beta^2-\omega^2)(\frac{1}{4}-\omega^2)^{\frac{1}{2}}}{D(\omega,\varphi)} \; 2i \; \text{Im} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_s)e^{-iR\sqrt{\frac{(\frac{1}{4}-\omega^2)}{(\beta^2-\omega^2)}(\omega^2-\beta^2\cos^2\varphi)}} \right\}$$

(4.48)

$$G_{II} \cong \frac{\cos\varphi}{2\pi R} \; \frac{(\omega^2-\beta^2)(\omega^2-\frac{1}{4})^{\frac{1}{2}}}{D(\omega,\varphi)} \; 2i \; \text{Re} \left\{ (\frac{2-\gamma}{2\gamma} + i\mu_s)e^{-iR\sqrt{\frac{(\omega^2-\frac{1}{4})}{(\omega^2-\beta^2)}(\omega^2-\beta^2\cos^2\varphi)}} \right\} .$$

(4.49)

Thus, the pressure in the far field can be written as integrals over the important frequency bands (cf. 4.28, 4.29):

$$e^{-\frac{1}{2}z}p_I \simeq \frac{\cos \varphi}{2\pi^2 R} \, \mathrm{Im} \int_{\beta \cos\varphi}^{\beta} \frac{\omega(\beta^2-\omega^2)(\frac{1}{4}-\omega^2)^{\frac{1}{2}}}{D(\omega,\varphi)} \left\{ \left(\frac{2-\gamma}{2\gamma} + i\mu_s\right)e^{i(\omega t+\Omega R)} \right.$$

$$\left. - \left(\frac{2-\gamma}{2\gamma} - i\mu_s\right)e^{i(\omega t-\Omega R)} \right\} d\omega \qquad (4.50)$$

$$e^{-\frac{1}{2}z}p_{II} \simeq \frac{\cos \varphi}{2\pi^2 R} \, \mathrm{Re} \int_{\frac{1}{2}}^{\infty} \frac{\omega(\omega^2-\beta^2)(\omega^2-\frac{1}{4})^{\frac{1}{2}}}{D(\omega,\varphi)} \left\{ \left(\frac{2-\gamma}{2\gamma} + i\mu_s\right)e^{i(\omega t-\Omega R)} \right.$$

$$\left. + \left(\frac{2-\gamma}{2\gamma} - i\mu_s\right)e^{i(\omega t+\Omega R)} \right\} d\omega \qquad (4.51)$$

where

$$\Omega(\omega,\varphi) = \sqrt{\frac{(\frac{1}{4}-\omega^2)}{(\beta^2-\omega^2)}(\omega^2-\beta^2\cos^2\varphi)} \, . \qquad (4.52)$$

The representation (4.50, 4.51) shows spherical waves with a spherical phase and group velocity depending on angle φ. These integrals can again be approximated by the method of stationary phase. Details are given in the next section.

In this section, we shall examine some of the properties of the pressure field given by (4.50) and (4.51). The field has been expressed in terms of integrals of the form

$$\int f(\omega, \cos \varphi) \, e^{i(\omega t \pm \Omega R)} \, d\omega \qquad (5.1)$$

over two bands of propagating frequencies, with $\Omega(\omega, \varphi)$ given by (4.52). For large R and t, the exponentials are rapidly oscillating and the method of stationary phase can be applied to these integrals. The points of stationary phase, if any, are given by

$$\frac{t}{R} \pm \frac{d\Omega}{d\omega} = 0 . \qquad (5.2)$$

The existence and number of stationary points in each frequency band is best illustrated graphically.

We first consider the high frequency band, $\frac{1}{2} \leq \omega < \infty$. In Fig. 5.1, we have plotted $\Omega(\omega, \varphi)$ vs ω for a typical value of φ. The function Ω is a monotonically increasing function of ω, starting out as $\sqrt{\omega - \frac{1}{2}}$ for ω near $\frac{1}{2}$ and asymptotically approaching ω as $\omega \to \infty$. The derivative $\frac{d\Omega}{d\omega}$ decreases continuously from ∞ to 1 as ω increases from $\frac{1}{2}$ to ∞. Since $\frac{d\Omega}{d\omega} > 0$, it is clear that only the exponential with the minus sign can have a stationary point. It is also clear from the above discussion that there will be a single stationary point in this frequency range for all t/R such that $1 \leq t/R < \infty$, i.e., for all points behind the spherical front.

For the low frequency band, $\beta \cos \leq \omega \leq \beta$, the situation is somewhat different, as can be seen from Fig. 5.2. The function Ω increases monotonically with ω in this band, starting out as $\sqrt{\omega - \beta \cos \varphi}$ for ω near $\beta \cos \varphi$ and approaching ∞ as $\dfrac{1}{\sqrt{\omega - \beta}}$ when ω
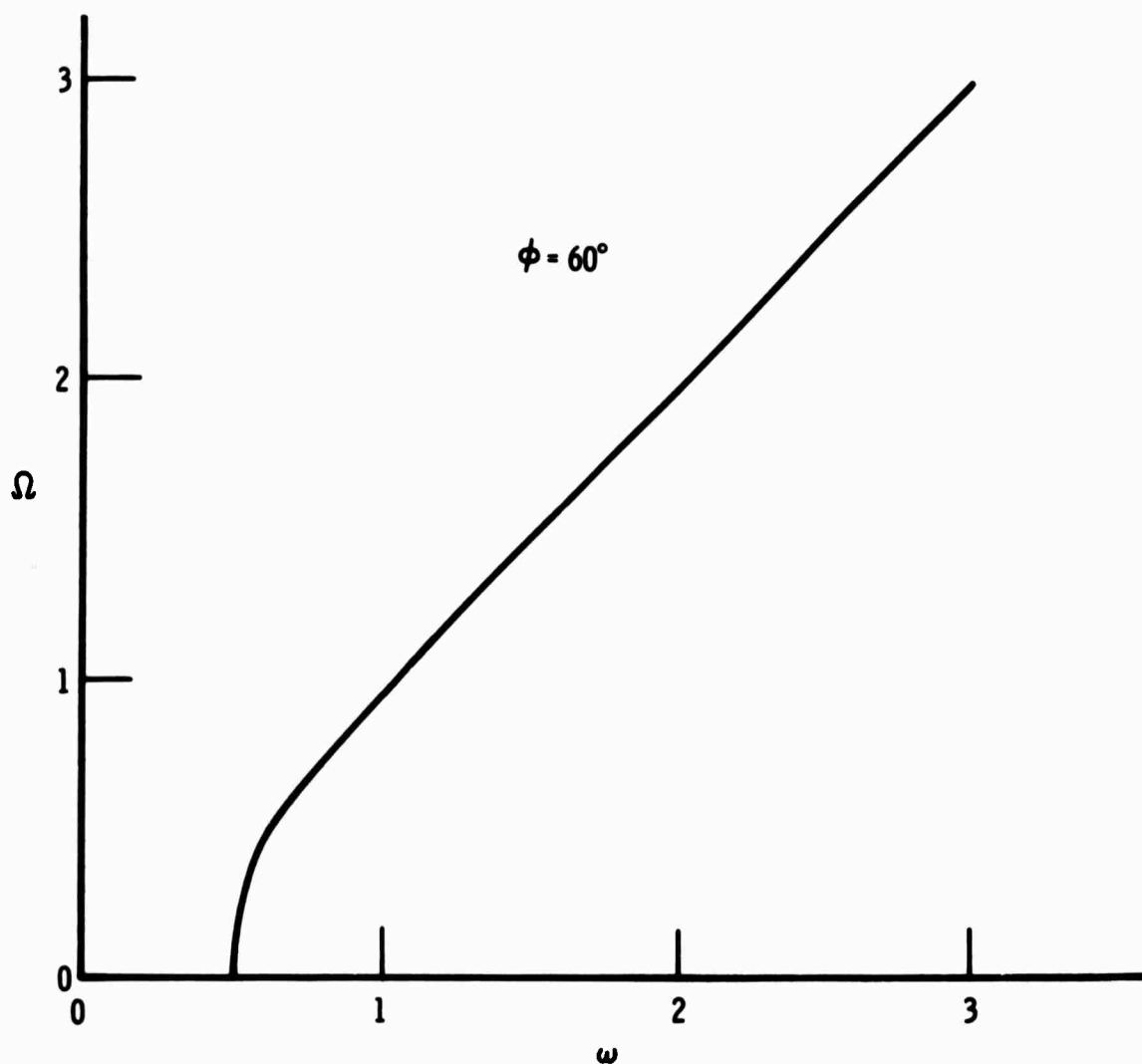
Fig. 5.1 - High frequency branch of the function $\Omega(\omega,\varphi)$ (Eq. 4.52) for $\varphi = 60°$. For a given ω, the slope of the curve is the value of t/R for which the phase is stationary at that frequency.

Fig. 5.2 - Low frequency branch of the function $\Omega(\omega,\varphi)$ (Eq. 4.52) for $\varphi = 60^\circ$. For a given ω, the slope of the curve is the value of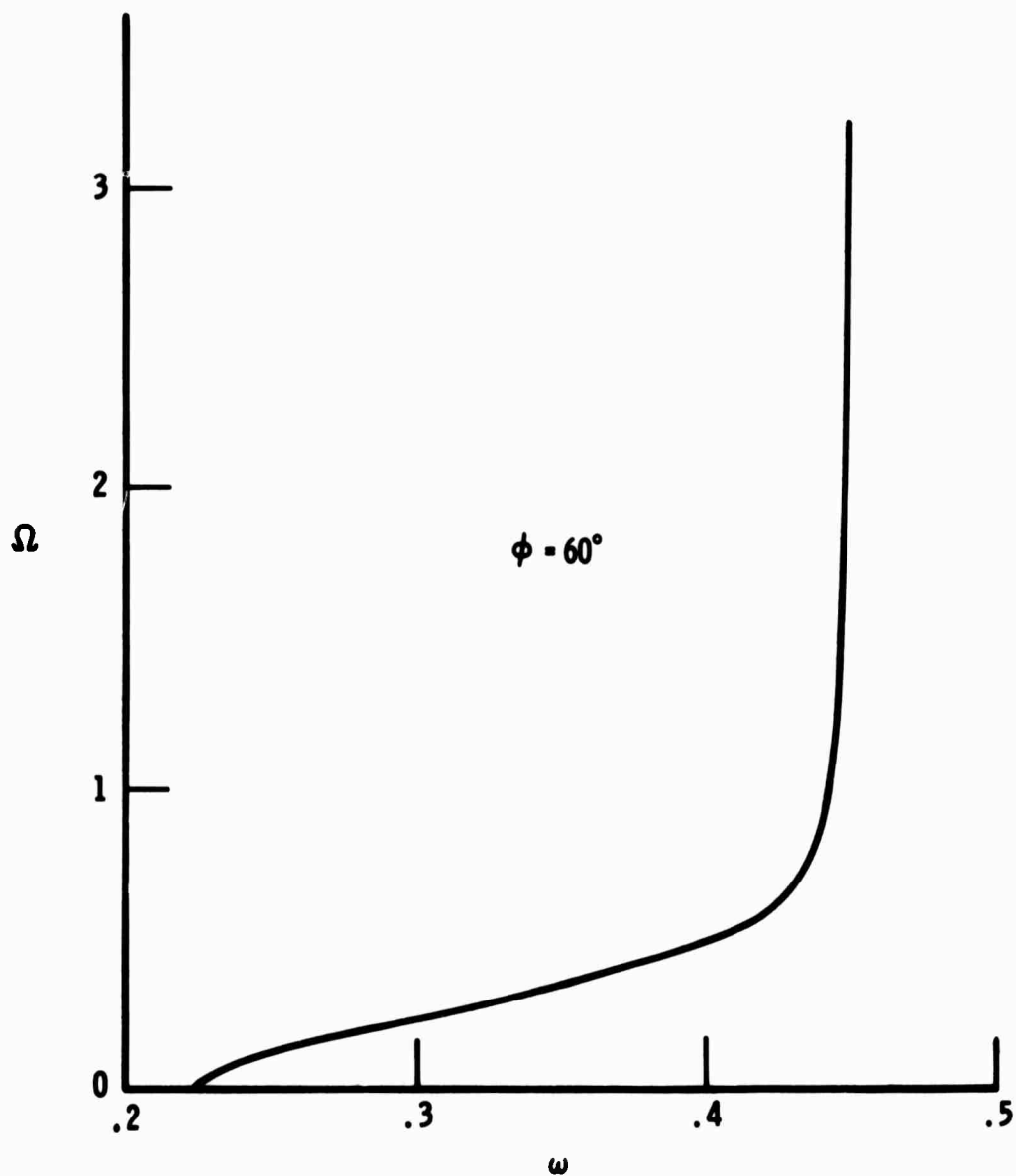 t/R for which the phase is stationary at that frequency. Note existence of inflection point, at which the slope has its minimum value.

approaches β. It is clear that the function has an inflection
point ω_c in this frequency range. The derivative $\frac{d\Omega}{d\omega}$ decreases from
∞ to some minimum value as ω increases from $\beta \cos \varphi$ to ω_c, and then
increases again to ∞ as ω increases from ω_c to β. For any given φ,
the minimum value of $\frac{d\Omega}{d\omega}$ represents the minimum value of t/R, or the
maximum value of R/t, for which a stationary point exists (again
only for the exponential with the minus sign). For values of R/t
greater than the maximum, there are no stationary points for the
given φ, whereas for values smaller than the maximum there are
clearly two stationary points. At the maximum value of R/t, the
two stationary points coincide.

This situation is identical to that which arises in the case of
incompressible flow in a density stratified liquid considered by
Mowbray and Rarity[4]. The physical consequences are the same as
for the incompressible case: the locus in physical space of the
double points of stationary phase defines a front, or "caustic,"
representing the onset of the disturbance. Between the caustic and
the spherical front, the low frequency part of the disturbance is
exponentially small, and the main contribution comes from the high
frequency band.

For the purposes of calculating the location of the front, it
is convenient to use the variables

$$\zeta = \frac{r}{t}$$
$$\eta = \frac{z}{t} \tag{5.3}$$

where $r = R \sin \varphi$ is the cylindrical radius and $z = R \cos \varphi$ is the
altitude. In terms of these variables, a line of constant phase
$\phi = \omega - \frac{R}{t} \Omega$ can be written

$$\frac{\omega^2}{(\beta^2-\omega^2)} \zeta^2 - \eta^2 = \frac{(\omega-\phi)^2}{(\xi-\omega^2)} \tag{5.4}$$

while the condition for stationary phase (5.2) becomes

$$\frac{\beta^2\omega(\xi-\omega^2)}{(\omega-\phi)(\beta^2-\omega^2)^2}\zeta^2 - \frac{\omega(\omega-\phi)}{(\xi-\omega^2)} = 1. \qquad (5.5)$$

For a given value of the parameter ϕ, which corresponds to a particular value of the phase at a given time, (5.4) and (5.5) are parametric equations for a line in the (ζ, η)-plane along which the particular phase is stationary. Each point on such a line corresponds to a different value of ω, the point of stationary phase. A line of constant and stationary phase intersects the locus of double stationary points in a cusp. The condition for a cusp is

$$\frac{d\zeta}{d\omega} = \frac{d\eta}{d\omega} = 0 . \qquad (5.6)$$

Applying this to (5.5), we can obtain ϕ as a function of ω along the front. Equations (5.4) and (5.5) then provide a one to one mapping from (ϕ, ω) to (ζ, η). The wave front obtained in this manner is shown in Fig. 5.3, where we have also plotted a few lines of constant ϕ for the purposes of illustration.

The co-ordinates (ζ, η) of any point on the front are the horizontal and vertical components, respectively, of the velocity of the front at that point. The point moving along the ground has a velocity $\zeta/t = 2\beta = .9c^*$. The maximum vertical velocity is $.26c^*$, and occurs at an angle of about 32° with the horizontal. (Every point on the front moves radially outward from the origin.)

As just indicated, each point on the caustic is associated with a given angle φ in physical space, given by $\varphi = \tan^{-1} \eta/\zeta$. Associated with this point on the caustic is also the frequency ω_c of the double point of stationary phase. This is the first frequency (in the low frequency band) to arrive at a given φ, after which the signal splits into two frequency components, one of which increases with time towards β and the other of which decreases towards $\beta \cos \varphi$. In Fig. 5.4, we plot in polar co-ordinates the frequency ω_c as a function of the pole angle φ.
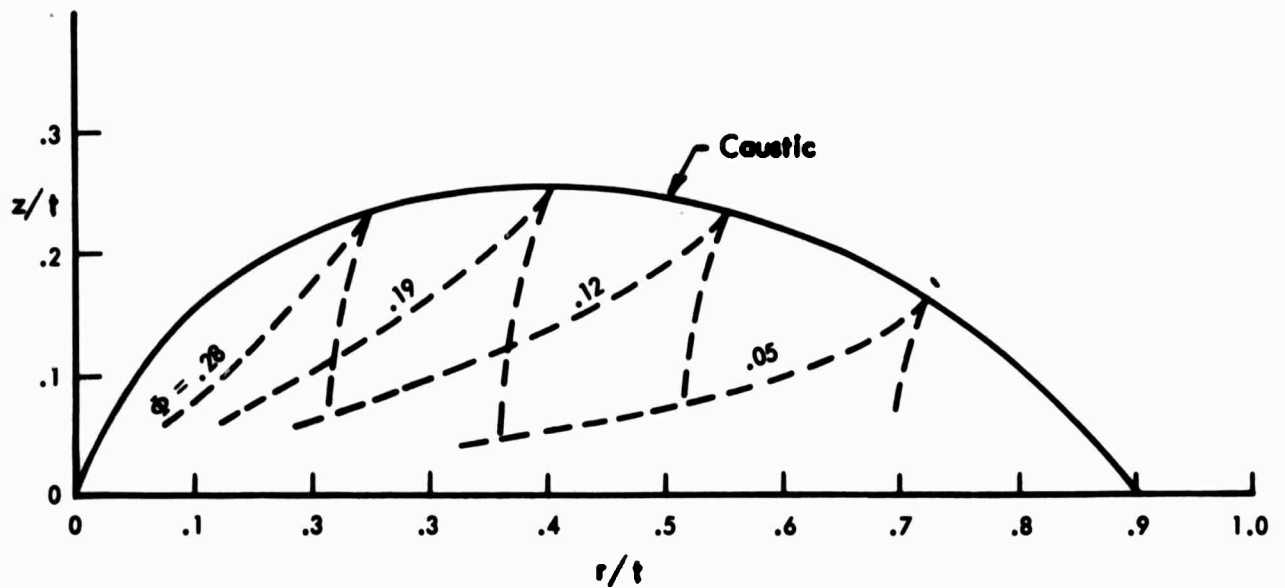
Fig. 5.3 - Location of the caustic in (r/t, z/t) space. The
dashed lines are lines of stationary phase along
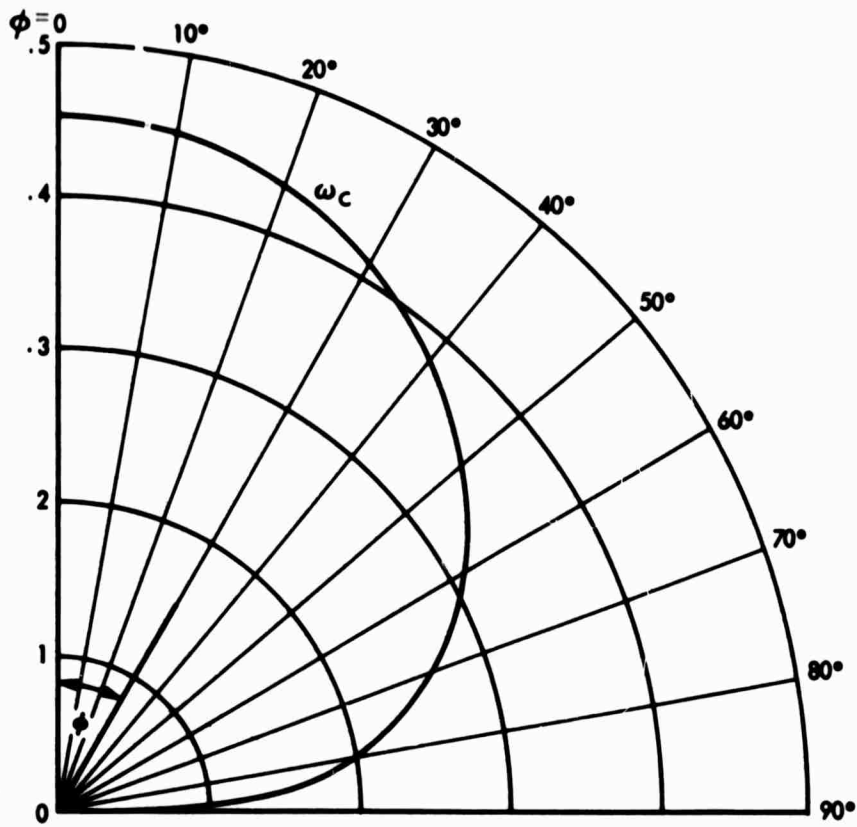which the phase is constant at any instant.

Fig. 5.4 - Angular frequency at the caustic, in units
of c*/h, as a function of the pole angle φ.

In light of the above discussion, it is now possible to
describe briefly the frequency-time history of the signal at any
location. The first disturbance to arrive at any location (R, φ) is
the acoustic spherical front, containing the very high frequencies.
As time progresses, the principal frequency in this part of the dis-
turbance decreases asymptotically towards $\omega = \frac{1}{2}$. Some time after
the passage of the spherical front, the caustic arrives, carrying
the frequency associated with the angle φ (Fig. 5.4). As dis-
cussed above, this part of the signal then splits into two frequen-
cies, one tending toward β and the other toward $\beta \cos \varphi$ as time
progresses. Thus, at any location, the signal consists asymptotically
of the three frequencies $\frac{1}{2}$, β, and $\beta \cos \varphi$. This information is
summarized in Fig. 5.5, which is a polar graph of the various
groups. Each dashed line represents the location in $(R/t, \emptyset)$ space
of a group of a given frequency. The solid line is the caustic,
which appears as the envelope of the low frequency band. Outside
of the caustic, only the high frequency band is present, whereas
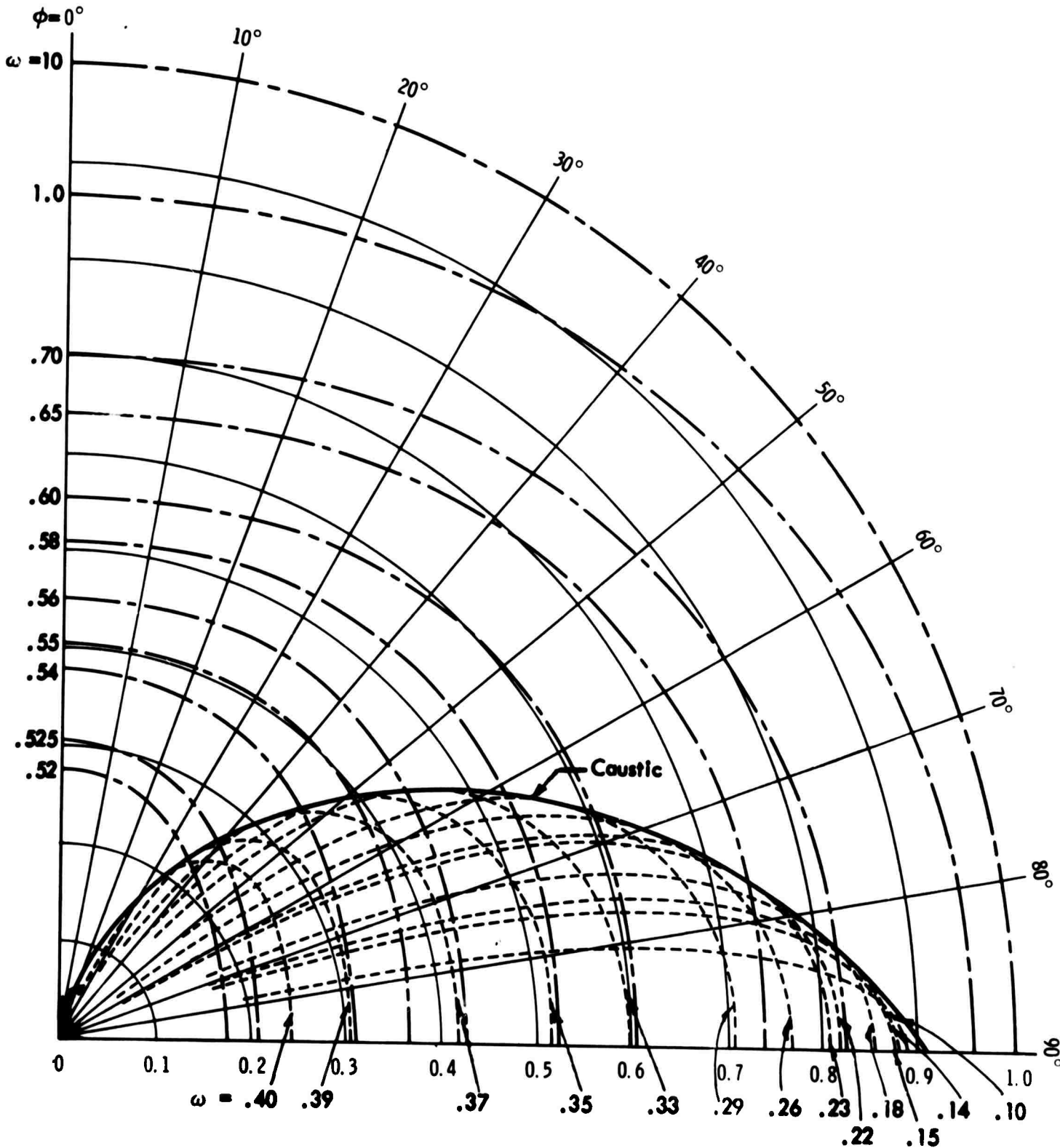inside of the caustic both bands clearly contribute to the signal.

Fig. 5.5 - Location in $(R/t, \varphi)$ space of groups of a given
frequency. The solid line is the caustic, which
appears as the envelope of the low frequency band.

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10. ABSTRACT

A complete asymptotic analysis is carried out for the flow field produced by the instantaneous release of energy, at a point on the ground, in an isothermal atmosphere. A double integral representation of the flow is constructed from Laplace-Handel transforms of the linearized equations. An asymptotic approximation to the integral is obtained by two successive applications of the method of stationary phase. It is found that there are three principal groups of dispersive waves behind the spherical acoustic front. One of these groups is contained in a high frequency band, and the other two are contained in a low frequency band. The spatial domain of the low frequency waves is cut off by a front (caustic), the location of which is calculated. Some of the characteristics of the flow are discussed.

11. KEY WORDS

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